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$L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$上のある種の自己同型写像について (作用素環の構造研究とその応用)

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$L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ 上のある種の自己同型写像について

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1. 序文

In this note we would like to explain the result of our paper [5]. In that paper we determine the structure of all automorphisms on $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ which preserve the subalgebra $L(SL(2,\mathbb{Z}))$ globally. The proof is a modification of the recent paper due to Neshveyev and Størmer for non-commutative groups. The uniqueness of HT-Cartan subalgebras due to Popa plays a crucial role in the proof.

The set of these automorphisms is denoted by $\mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))$, and we write

$$\mathrm{Int}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z}))) = \{\mathrm{Ad}w : \ w \text{ is a unitary in } L(SL(2,\mathbb{Z}))\}.$$

Our main result is

$$\mathrm{Out}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z}))) \simeq \mathbb{Z}_{12} \rtimes \mathbb{Z}_2,$$

where

$$\mathrm{Out}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))$$

$$= \mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))/\mathrm{Int}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))$$

and $\mathbb{Z}_2$ acts on $\mathbb{Z}_{12}$ by the inverse operation. Indeed the automorphism group $\mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z})))$ can be completely described by the irreducible characters and automorphisms on $SL(2,\mathbb{Z})$.

2. 主結果

The unimodular group $SL(2,\mathbb{Z})$ acts on $\mathbb{Z}^2$ by the matrix multiplication. Then its dual action on $\hat{\mathbb{Z}}^2 = \mathbb{T}^2$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot (z,w) = (z^a w^c, z^b w^d)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. We shall freely identify these two actions via the Fourier transformation and this identification induces the natural isomorphism between $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ and $L^\infty(\mathbb{T}^2) \rtimes_\alpha SL(2,\mathbb{Z})$, where α denotes the action of $SL(2,\mathbb{Z})$ on $L^\infty(\mathbb{T}^2)$ induced by this action.

For each automorphism β on $SL(2,\mathbb{Z})$, consider all measure-preserving transformations S on $\mathbb{T}^2$ such that $Sg = \beta(g)S$ for $g \in SL(2,\mathbb{Z})$. We denote by I_β the set consisting of these type transformations. A measure-preserving transformation T on $\mathbb{T}^2$ induces the automorphism σ_T defined by $\sigma_T(f)(x) = f \circ T^{-1}(x)$ ($f \in L^\infty(\mathbb{T}^2)$, $x \in \mathbb{T}^2$). For $S \in I_\beta$, the automorphism σ_S can be extended to

$L^\infty(\mathbb{T}^2) \rtimes_\alpha SL(2,\mathbb{Z})$ by $\sigma_S(\lambda_g) = \lambda_{\beta(g)}$, where λ_g is the canonical implementing unitary. An irreducible character χ on $SL(2,\mathbb{Z})$ also gives the automorphism σ_χ on $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ such that $\sigma_\chi(\lambda_g) = \chi(g)\lambda_g$ and $\sigma_\chi|_{L(\mathbb{Z}^2)} = \mathrm{id}$.

The following theorem is an analogue of [8] Theorem4.2 for the non-commutative group $SL(2,\mathbb{Z})$. We would like to emphasize that in the original proof [8] the commutativity of groups plays a crucial role. Thus we need some more effort to prove the theorem.

Theorem 2.1. *Let γ be an automorphism on $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ satisfying $\gamma(L(SL(2,\mathbb{Z}))) = L(SL(2,\mathbb{Z}))$. Then there exist a unitary $w \in L(SL(2,\mathbb{Z}))$, an irreducible character χ on $SL(2,\mathbb{Z})$, an automorphism β on $SL(2,\mathbb{Z})$ and a transformation $S \in I_\beta$ such that*

$$\gamma = \mathrm{Ad}w\sigma_S\sigma_\chi.$$

This theorem enables us to determine the structure of $\mathrm{Out}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(S$

as follows.

Corollary 2.2. *We have an isomorphism*

$$\mathrm{Out}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})), L(SL(2,\mathbb{Z}))) \simeq \mathbb{Z}_{12} \rtimes \mathbb{Z}_2.$$

Proof of Corollary 2.2. First we shall show that $\sigma_S \sigma_\chi$ is an outer automorphism on $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ whenever β is outer or χ is a non-trivial character. In order to show this fact, we need the following claim:

<u>Claim</u>

$L(SL(2,\mathbb{Z}))$ is singular in $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$, i.e., if w is a normalizer of $L(SL(2,\mathbb{Z}))$ in $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$, then w must belong to $L(SL(2,\mathbb{Z}))$.

The proof of this claim will be postponed until the end of this section.

If $\sigma_S\sigma_\chi = \mathrm{Ad}w$ for some unitary $w \in L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$, then w is a normalizer of $L(SL(2,\mathbb{Z}))$ and hence $w \in L(SL(2,\mathbb{Z}))$ by the above claim. Then by the proof of [8] Proposition 2.2, we have $w = c\lambda_g$ for some scalar c and $g \in SL(2,\mathbb{Z})$. (Indeed this can be easily seen by using the Fourier expansion of w.) Then the direct computations show that this can be occur only when β is inner and χ is trivial.

Thanks to the above consideration, we have only to prove that the subgroup generated by $\{\sigma_S\}_{S\in I_\beta,\beta}$ and $\{\sigma_\chi\}_\chi$ in $\mathrm{Aut}(L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})))$ is isomorphic to $\mathbb{Z}_{12} \rtimes \mathbb{Z}_2$.

It is a well-known fact that $SL(2,\mathbb{Z}) \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ where

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

are generators of $\mathbb{Z}_4$, $\mathbb{Z}_6$ and $\mathbb{Z}_2$ respectively. Hence all irreducible characters on $SL(2,\mathbb{Z})$ are of the form $\chi_1 * \chi_2$ for some $\chi_1 \in \hat{\mathbb{Z}}_4$ and $\chi_2 \in \hat{\mathbb{Z}}_6$ which coincide on

$\mathbb{Z}_2$. Thus it is easily seen that the group consisting of all irreducible characters on $SL(2,\mathbb{Z})$ is isomorphic to $\mathbb{Z}_{12}$.

It is also well-known that up to inner automorphism, the map $\beta = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the unique outer automorphism on $SL(2,\mathbb{Z})$ which does not come from some character([6]). Clearly this map β induces the inverse operation on $\mathbb{Z}_4, \mathbb{Z}_6 (\subset SL(2,\mathbb{Z}))$ and hence on the characters. Define the transformation S on $\mathbb{T}^2$ by $S(z,w) = (z,\overline{w})$. Then the direct computations show that $S \in I_\beta$. Note that S (and σ_S) has period 2 and $\sigma_S\sigma_\chi = \sigma_{\chi\circ\beta}\sigma_S$ holds. In [3] Golodets showed that I_{id} consists of exactly two elements; identity map and conjugation map. It is easily seen that $S_1^{-1} \cdot S_2 \in I_{id}$ if $S_1, S_2 \in I_\beta$. Since the conjugation map is given by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (the generator of $\mathbb{Z}_2$), we have the above statement. □

In order to prove the above theorem, we need the uniqueness theorem for HT-Cartan subalgebras due to Popa. More precisely, we need:

Theorem 2.3 ([10] 4.1. Theorem.). *Let γ be an automorphism on $L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ satisfying $\gamma(L(SL(2,\mathbb{Z}))) = L(SL(2,\mathbb{Z}))$. Then there exists a unitary $u \in L(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ such that* $\mathrm{Ad}u^*\gamma(L(\mathbb{Z}^2)) = L(\mathbb{Z}^2)$.

Thanks to this theorem, we are now in the same situation as that of [8]. Unfortunately Neshveyev and Størmer's proof uses the commutativity of the group frequently, so we cannot apply their argument directly. However their argument does work in our setting with some modifications.

The rest of this section will be devoted to the proof of the main result. Our strategy is very much simple, which is a modification of the argument in the paper due to Neshveyev and Størmer ([8]) in the non-commutative group setting.

We consider the standard representation of $L^\infty(\mathbb{T}^2) \rtimes_\alpha SL(2,\mathbb{Z})$ on $L^2(\mathbb{T}^2) \otimes l^2(SL(2,\mathbb{Z}))$. Let π be the representation of $L^\infty(\mathbb{T}^2)$ given by

$$\pi(f) = \sum_{g \in SL(2,\mathbb{Z})} \alpha_{g^{-1}}(f) \otimes e_g,$$

where e_g is the minimal projection on $\mathbb{C}\delta_g$ ($g \in SL(2,\mathbb{Z})$) and $f \in L^\infty(\mathbb{T}^2)$. We sometimes omit the symbol π, so the reader should not confuse $\pi(L^\infty(\mathbb{T}^2))$ with $L^\infty(\mathbb{T}^2) \otimes I$. We denote the left regular representation (resp. the right regular antirepresentation) of $SL(2,\mathbb{Z})$ on $l^2(SL(2,\mathbb{Z}))$ by λ_g (resp. ρ_g) ($g \in SL(2,\mathbb{Z})$). Thus $L(SL(2,\mathbb{Z})) = \{\lambda_g\}''_{g \in SL(2,\mathbb{Z})}$ and $L^\infty(\mathbb{T}^2) \rtimes_\alpha SL(2,\mathbb{Z})$ is generated by $\pi(L^\infty(\mathbb{T}^2))$ and $L(SL(2,\mathbb{Z}))$.

The algebras $L^\infty(\mathbb{T}^2)$ and $L(SL(2,\mathbb{Z}))$ act standardly on $L^2(\mathbb{T}^2)$ and $l^2(SL(2,\mathbb{Z}))$ respectively. For each automorphism $\alpha \in \mathrm{Aut}(L^\infty(\mathbb{T}^2))$ (resp. $\alpha' \in \mathrm{Aut}(L(SL(2,\mathbb{Z})))$), we denote its canonical implementing unitary by $u_\alpha \in B(L^2(\mathbb{T}^2))$ (resp. $v_{\alpha'} \in$

$B(l^2(SL(2,\mathbb{Z}))))$. For each measure-preserving transformation S on $\mathbb{T}^2$, we write $u_S = u_{\sigma_S}$. We also use the notation $u_g = u_{\alpha_g}$, $v_\chi = v_{\sigma_\chi}$ and $v_\beta = v_{\sigma_S}$ $(S \in I_\beta)$.

The modular conjugation of $L^\infty(\mathbb{T}^2) \rtimes_\alpha SL(2,\mathbb{Z})$ is denoted by J. It is easily seen that $J\pi(f)J = \overline{f} \otimes I$ and $J(I \otimes \lambda_g)J = u_g \otimes \rho_g^*$.

Let γ be as in the theorem and take a unitary $u \in L^\infty(\mathbb{T}^2) \rtimes_\alpha SL(2,\mathbb{Z})$ such that $u^*\gamma(L^\infty(\mathbb{T}^2))u = L^\infty(\mathbb{T}^2)$ (Here we use the uniqueness of HT-Cartan-subalgebras). Let $\tilde{\gamma} = \mathrm{Ad}u^*\gamma$. The canonical implementation of γ is given by U_γ and we define $U = Ju^*JU_\gamma$. Then it is easily seen that $\mathrm{Ad}U|_{L^\infty(\mathbb{T}^2)\rtimes_\alpha SL(2,\mathbb{Z})} = \gamma$ and $\mathrm{Ad}U|_{L^\infty(\mathbb{T}^2)\otimes I} = \tilde{\gamma}$. (Remark that in $L^\infty(\mathbb{T}^2) \rtimes_\alpha SL(2,\mathbb{Z})$, $\tilde{\gamma}$ preserve $\pi(L^\infty(\mathbb{T}^2))$ globally. Hence we can define the automorphism $\tilde{\gamma}\otimes I$ on $L^\infty(\mathbb{T}^2)\otimes I$). Define $W = U(u_{\tilde{\gamma}}^* \otimes v_\gamma^*)$. Clearly W belongs to $L^\infty(\mathbb{T}^2) \otimes R(SL(2,\mathbb{Z}))$. (See [8], Consider the Fourier expansion

$$\tilde{\gamma}^{-1}(\lambda_h) = \sum_{g \in SL(2,\mathbb{Z})} E(\tilde{\gamma}^{-1}(\lambda_h)\lambda_g^*)\lambda_g,$$

where E denotes the trace-preserving conditional expectation on $\pi(L^\infty(\mathbb{T}^2))$. Let $f_g^{(h)}$ be the support projection of $E(\tilde{\gamma}^{-1}(\lambda_h)\lambda_g^*)$. The next lemma is obvious.

Lemma 2.4. $f_g^{(h)} \perp f_{g'}^{(h)}$ $(g \neq g')$. $\sum_{g \in SL(2,\mathbb{Z})} f_g^{(h)} = I$.

For almost all $x \in \mathbb{T}^2$, there exists the unique element $g(h,x) \in SL(2,\mathbb{Z})$ such that $f_{g(h,x)}^{(h)}(x) = 1$ and $f_g^{(h)}(x) = 0$ $(g \neq g(h,x))$. Define $\tilde{g}(h,x) = g(h,\sigma^{-1}x)$ where σ is a measure-presearving transformation corresponding to $\tilde{\gamma}$, i.e., σ satisfies $\tilde{\gamma}(f) = f \circ \sigma^{-1}$ for $f \in L^\infty(\mathbb{T}^2)$.

The following lemma is well-known.

Lemma 2.5. *For almost all $x \in \mathbb{T}^2$, we have $g(h,x)^{-1}x = \sigma^{-1}h^{-1}\sigma x$. The map $g(h,x)$ is a 1-cocycle with respect to $\tilde{\gamma}^{-1}\alpha\tilde{\gamma}$, i.e., $g(h,x)g(k,\sigma^{-1}h^{-1}\sigma x) = g(hk,x)$. (Hence $\tilde{g}(h,x)$ is a 1-cocycle with respect to α.)*

The automorphism γ is extended to $R(SL(2,\mathbb{Z}))$ by $\mathrm{Ad}v_\gamma$.

By using the Fourier expansion of $\tilde{\gamma}^{-1}(\lambda_h)$ and Lemma 2.4, we can show the next lemma.

Lemma 2.6. $W(h^{-1}x) = t(h,x)\rho_h W(x)\gamma(\rho^*_{\tilde{g}(h,x)})$, *where* $t(h,x) = E(\tilde{\gamma}(\lambda_{\tilde{g}(h,x)})\lambda^*_h)(x)$.

From Lemma 2.6, we can easily show the following.

Lemma 2.7. *Denote the comultiplication on $R(SL(2,\mathbb{Z}))$ by Δ, which is defined as $\Delta(\rho_g) = \rho_g \otimes \rho_g$. Then we have*

$$F(h^{-1}x) = t(h,x)\Phi(h)^*F(x)\Psi(h),$$

where F, Φ and Ψ are defined by $F(x) = \gamma^{-1}(W(x)) \otimes \gamma^{-1}(W(x)) \Delta \circ \gamma^{-1}(W(x))^*$, $\Phi(h) = \gamma^{-1}(\rho_h)^* \otimes \gamma^{-1}(\rho_h)^*$ *and* $\Psi(h) = \Delta \circ \gamma^{-1}(\rho_h)^*$. *Note that Φ and Ψ are unitary representation of $SL(2,\mathbb{Z})$.*

We will use the following well-known fact: there is a sequence $\{h_n\}_{n=1}^{\infty} \subset SL(2,\mathbb{Z})$ which has the properties (1) h_n tends to infinity, (2) for any finite set $\Omega \subset \mathbb{Z}^2$ such that $(0,0) \notin \Omega$, we can find a sufficiently large n_0 such that $h_n\Omega \cap \Omega = \emptyset$ for $n > n_0$. Indeed if we let for example

$$h_n = \begin{pmatrix} n^2 - n + 1 & n \\ n - 1 & 1 \end{pmatrix},$$

then it is easy to see that this sequence h_n is the desired one.

Take such $\{h_n\}_{n=1}^{\infty} \subset SL(2,\mathbb{Z})$ and fix it. Recall that the unitary F belongs to $L^{\infty}(\mathbb{T}^2) \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$. The unitaries $\Phi(h)$ and $\Psi(h)$ $(h \in SL(2,\mathbb{Z}))$ belong to $I \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$. Let $z_h(x) = t(h,x)$. Then z_h is an unitary element in $L^\infty(\mathbb{T}^2)$. The previous lemma means that

$$\alpha_h(F) = z_h\Phi(h)^*F\Psi(h).$$

Lemma 2.8. *The automorphism* $\theta = \mathrm{Ad}F$ *on* $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ *satisfies*

$$\theta(I \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))) = I \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z})).$$

Proof. By using the previous lemma, it is easily seen that

$$FxF^* = \Phi(h)\alpha_h(F)\Psi(h)^*x\Psi(h)\alpha_h(F)^*\Phi(h)^*$$

for any $x \in I \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$ $(||x|| \leq 1)$. In particular this equality holds for h_n. For any $\epsilon > 0$, we can replace F by F_0 such that it has the finite support as an element of $L(\mathbb{Z}^2)\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$. That is, by Kaplansy density theorem, there exists a finite subset $\Omega \subset \mathbb{Z}^2$ such that $F_0 = \sum_{g\in\Omega} a_g\delta_g$ (a_g is an element of $R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$) and $||F - F_0||_2 < \epsilon$ and $||F_0|| \leq ||F||$.

Hence we get

$$||F_0 x F_0^* - \Phi(h)\alpha_h(F_0)\Psi(h)^* x \Psi(h)\alpha_h(F_0)^*\Phi(h)^*|| < 4\epsilon.$$

As n goes to infinity, the support of $\Phi(h)\alpha_h(F_0)\Psi(h)^* x \Psi(h)\alpha_h(F_0)^*\Phi(h)^*$ goes to infinity except for the unit $(0,0) \in \mathbb{Z}^2$, while the support of $F_0 x F_0^*$ does not change. Since ϵ is arbitrary, this means that the support of FxF^* consists of only one point $(0,0)$. Hence $FxF^* \in I \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$.

By the same argument, we can also see that $F^*xF \in I \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$. Thus we get the statement. □

Let $\Pi(h) = F\Psi(h)^*F^*\Phi(h)$. Then we have $\alpha_h(F) = z_h\Pi(h)^*F$. Thanks to the previous lemma, each $\Pi(h)$ belongs to $I \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$.

Lemma 2.9. (i) *z_h is an α-one cocycle.*

(ii) *Π is an unitary representation of $SL(2,\mathbb{Z})$.*

Proof. Obvious. □

Since $F\Psi(h)F^*\Pi(h) = \Phi(h)$ and $F\Psi(h)F^*$, $\Pi(h)$, $\Phi(h)$ are representations, we have $F\Psi(h)F^*\Pi(k) = \Pi(k)F\Psi(h)F^*$. Indeed we have

$$F\Psi(h)F^*F\Psi(k)F^*\Pi(h)\Pi(k) = \Phi(hk) = \Phi(h)\Phi(k)$$

$$= F\Psi(h)F^*\Pi(h)F\Psi(k)F^*\Pi(k).$$

Hence $F\Psi(k)F^*\Pi(h) = \Pi(h)F\Psi(k)F^*$.

Lemma 2.10. *The von Neumann algebra generated by $\Pi(SL(2,\mathbb{Z}))$ is finite dimensional.*

Proof. As noted above, we know that $\Pi(SL(2,\mathbb{Z})) \subset (F\Psi(SL(2,\mathbb{Z}))F^*)' \cap (R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z})))$. Since $\mathrm{Ad}F$ preserves $R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$ globally, it is enough to show that $(\Psi(SL(2,\mathbb{Z})))' \cap (R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z})))$ is finite dimensional. Recall that $\Psi(SL(2,\mathbb{Z}))'' = \Delta(SL(2,\mathbb{Z}))'' = \{g \otimes g : \ g \in SL(2,\mathbb{Z})\}''$.

Combining this with the fact $SL(2,\mathbb{Z}) \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$, it is easy to see that $(\Psi(SL(2,\mathbb{Z})))' (R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z})))$ is 4-dimensional. $\square$

Lemma 2.11. *There exist unitaries $z \in L^\infty(\mathbb{T}^2)$ and $A \in R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ such that $F = z^* \otimes A$.*

Proof. Since Π is a finite-dimensional representation, we may assume that $\Pi(h_n)$ converges to a unitary $X \in R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$ in the norm topology. Hence

$$||\alpha_{h_n}(F) - z_{h_n} X F||$$

converges to zero as $n \to \infty$. Take a spectral projection e of X such that $eX = wX$ where $w \in \mathbb{T}$. Since e is a fixed point of α (because $e \in R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$), we get

$$||\alpha_{h_n}(eF) - z_{h_n} w(eF)||$$

converges to zero. For each normal state ρ on $R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$, we denote by T_ρ the slice map from $L^\infty(\mathbb{T}^2)\otimes R(SL(2,\mathbb{Z}))\otimes R(SL(2,\mathbb{Z}))$ onto $L^\infty(\mathbb{T}^2)$, i.e., $T_\rho(x \otimes y) = \rho(y)x$ for $x \in L^\infty(\mathbb{T}^2)$ and $y \in R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$. Obviously T_ρ commutes with α. Hence

$$||\alpha_{h_n}(T_\rho(eF)) - z_{h_n}w(T_\rho(eF))||$$

converges to zero. Since eF is a non-zero element (because F is unitary), we can choose ρ such that $f = T_\rho(eF)$ is also non-zero. Next we claim that $g = |f|$ is a constant function. Indeed, since

$$||\alpha_{h_n}(f) - z_{h_n}wf||$$

converges to zero, $||\alpha_{h_n}(g) - g||$ also converges to zero. As in the proof of Lemma 2.8, by comparing their supports as elements of $L(\mathbb{Z}^2)$, we conclude that g is constant. Thus we may assume that f is unitary. Since both $||\alpha_{h_n}(F) - z_{h_n}XF||$ and $||\alpha_{h_n}(f) - z_{h_n}wf||$ converge to zero, $||\alpha_{h_n}(F) - (\overline{w}f^*\alpha_{h_n}(f))XF||$ and hence

$||\alpha_{h_n}(f^*F) - \overline{w}X(f^*F)||$ converge to zero. Considering the supports again, we see that f^*F is a (operator-valued) constant function, i.e., $f^*F \in I \otimes R(SL(2,\mathbb{Z})) \otimes R(SL(2,\mathbb{Z}))$. This means that F is of the desired form. $\square$

Combining this lemma with $\alpha_h(F) = z_h\Phi(h)^*F\Psi(h)$, we get

$$A^*\Phi(h)A = \frac{z(h^{-1}x)t(h,x)}{z(x)}\Psi(h).$$

This implies that the map $h \mapsto \frac{z(h^{-1}x)t(h,x)}{z(x)}$ is independent of the choice of x almost everywhere and define the irreducible character χ on $SL(2,\mathbb{Z})$. Hence we have $t(h,x) = \frac{z(x)\chi(h)}{z(h^{-1}x)}$.

Therefore if we replace u by uz, we may assume that $t(h,x) = \chi(h)$, $F(h^{-1}x) = \chi(h)\Phi(h)^*F(x)\Psi(h)$ for almost all $x, y \in \mathbb{T}^2$. Indeed, we have

$$E(\tilde{\gamma}(\lambda_{\tilde{g}(h,x)})\lambda_h^*)(x) = t(h,x) = \frac{z(x)}{z(h^{-1}x)}\chi(h)$$

and hence

$$E(\tilde{\gamma}(\lambda_g)\lambda_h^*) = z\alpha_h(z)^*\chi(h)\tilde{\gamma}(f_g^{(h)}).$$

$$E(z^*u^*\gamma(\lambda_g)uz\lambda_h^*) = \chi(h)\tilde{\gamma}(f_g^{(h)}).$$

Of course uz satisfies $(uz)^*L^\infty(\mathbb{T}^2)(uz) = L^\infty(\mathbb{T}^2)$. Hence we may assume that $t(h,x) = \chi(h)$ and $F(h^{-1}x) = \chi(h)\Phi(h)^*F(x)\Psi(h)$ for almost all $x, y \in \mathbb{T}^2$. From this equation, the same argument as in the proof of Lemma 2.8 shows the following.

Lemma 2.12. *$F(x) = F(y)$ for almost all $x, y \in \mathbb{T}^2$.*

Lemma 2.13. *There exist a unitary $w_0 \in R(SL(2,\mathbb{Z}))$, an automorphism β on $SL(2,\mathbb{Z})$ and the map $\mathbb{T}^2 \ni x \mapsto g(x) \in SL(2,\mathbb{Z})$ such that $\tilde{g}(h,x) = g(x)\beta^{-1}(h)g(h^{-1}x)^{-1}$ and $\gamma(\rho_g) = \chi(g)w_0^*\rho_{\beta(g)}w_0$.*

Proof. Since F is a (operator-valued) constant function, we have

$$\gamma^{-1}(W(x))\otimes\gamma^{-1}(W(x))\Delta\circ\gamma^{-1}(W(x))^* = \gamma^{-1}(W(y))\otimes\gamma^{-1}(W(y))\Delta\circ\gamma^{-1}(W(y))^*.$$

By letting $F(x,y)=\gamma^{-1}(W(y)^*W(x))$, we get

$$F(x,y)\otimes F(x,y)=\Delta(F(x,y)).$$

This implies that for almost all $x,y\in\mathbb{T}^2$, we can find the unique $g(x,y)\in SL(2,\mathbb{Z})$ such that $F(x,y)=\rho_{g(x,y)}$. Fix $x_0\in\mathbb{T}^2$ and let $w_0=W(x_0)$, $g(x)=g(x,x_0)$. We then have

$$\gamma^{-1}(w_0^*W(x))=F(x_0,x)=\rho_{g(x_0,x)}=\rho_{g(x)}$$

and hence $W(x)=w_0\gamma(\rho_{g(x)})$. Combining this with $W(h^{-1}x)=\chi(h)\rho_h W(x)\gamma(\rho^*_{\tilde{g}(h,x)})$, we get

$$\gamma^{-1}\circ \mathrm{Ad}w_0^*(\rho_h)=\chi(h^{-1})\rho_{g(x)^{-1}\tilde{g}(h,x)g(h^{-1}x)}.$$

From this equation, we can find an automorphism β on $SL(2,\mathbb{Z})$ such that $\tilde{g}(h,x)=g(x)\beta^{-1}(h)g(h^{-1}x)^{-1}$ and $\gamma(\rho_g)=\chi\circ\beta(g)w_0^*\rho_{\beta(g)}w_0$. $\square$

The rest of the proof is completely same as that of [8]. Hence we would like to

Finally we would like to show the claim stated in the proof of Corollary 2.2. The proof is essentially same as that of [8] Theorem 2.1. However, since we are dealing with the non-commutative group $SL(2,\mathbb{Z})$, in order to prove the claim we need the triviality of "operator-valued eigenfunctions" on $\mathbb{T}^2$. We have already used this type argument in the proof of Lemmas 2.8, 2.11 and 2.12.

Proof. (Proof of the claim which we have postponed) Let w be a normalizer of $L(SL(2,\mathbb{Z}))$. Define $\theta = \mathrm{Ad}w$ and $v = w(I \otimes v_\theta^*)$. Note that $v \in L^\infty(\mathbb{T}^2) \otimes R(SL(2,\mathbb{Z}))$. Compute

$$v(I \otimes v_\theta) = w = J\lambda_h J w J \lambda_h^* J$$

$$= (u_h \otimes \rho_h^*) w (u_h^* \otimes \rho_h)$$

Hence we get $\alpha_h(v)(I \otimes \rho_h^* \theta(\rho_h)) = v$. Then the same argument in the proof of Lemma 2.8 shows that $v \in R(SL(2,\mathbb{Z}))$. Thus $w = v(I \otimes v_\theta) \in I \otimes B(l^2(SL(2,\mathbb{Z})))$.

Combining this with the fact that w commutes with $J\lambda_h J = u_h \otimes \rho_h^*$, we see that

w must belong to $L(SL(2,\mathbb{Z}))$. □

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